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THE MEANING OF ROTATION IN THE SPECIAL THEORY OF RELATIVITY

BY PHILIP FRANKLIN

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

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The novelty of the special theory of relativity consists in its replacing the equations for translation given by the Newtonian theory by the Lorentz transformations. It is natural to ask what new equations should be used to express a rotation in this theory. Most of the writers on the subject (*e. g.*, A. Einstein, *Leipzig, Ann. Physik*, **49,** 1916, §3; M. V. Laue, *Die Relativitätstheorie*, Vol. 2, 1921, p. 162) have assumed that the Newtonian equations for rotation may be used without change in the new theory. While this is permissible as a "first approximation" for points near the axis of rotation, it is not a satisfactory transformation for the whole of space, since it involves velocities greater than that of light for points sufficiently far from the axis. The object of this note is to present a definition of rotation which is free from this objection, and to deduce some of its properties.

In the Newtonian theory the equations of transformation connecting the coördinates in a system K with those in a system K' moving uniformly with respect to K along the x-axis are:

$$x' = x - vt, y' = y, z' = z, t' = t. \tag{1}$$

For a rotation about the z-axis, we have:

$$x' = x \cos\omega t + y \sin\omega t, y' = -x \sin\omega t + y \cos\omega t, z' = z, t' = t, \tag{2}$$

or if polar coördinates are introduced in the xy plane:

$$\Theta' = \Theta - \omega t, r' = r, z' = z, t' = t. \tag{2a}$$

In the special theory of relativity, in place of (1) we have the Lorentz transformations:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, y' = y, z' = z, t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}. \qquad (3)$$

To obtain the corresponding generalization of (2a) we shall assume the following properties, which may be used to derive (2a) from (1), as characteristic of uniform rotation.

1. The velocity of a point in K' with respect to the point in K with which it momentarily coincides is independent of the time, and is the same for all points at a given distance from the axis of rotation.

2. For the two concentric circles $r' = r =$ const., the equations of transformation are similar to those for a uniform translation with $r\Theta$ the length of the arc along the circle in place of the linear distance x.

3. The velocity of a point at the distance $r' + \Delta r'$ from the axis with respect to a point at the distance r' from the axis (both in the system K') is a constant (ω) times $\Delta r'$ where $\Delta r'$ is a differential.

From 1 and 2 we find:

$$r'\Theta' = \frac{r\Theta - v(r).t}{\sqrt{1 - v(r)^2/c^2}}, \ r' = r, z' = z, t' = \frac{t - v(r).r.\Theta/c^2}{\sqrt{1 - v(r)^2/c^2}}. \qquad (4)$$

To utilize 3, we recall the equation for compounding velocities, obtained by differentiating (3): $(dx'/dt' = v_{12};\ dx/dt = v_2;\ v = v_1)$

$$v_{12} = \frac{v_2 - v_1}{1 - v_1 v_2/c^2}. \qquad (5)$$

For our case we have $v_2 = v(r + \Delta r);\ v_1 = v(r);\ v_{12} = \omega\Delta r$, giving:

$$\omega\Delta r = \frac{v(r + \Delta r) - v(r)}{1 - v(r + \Delta r)\, v(r)/c^2}, \qquad (6)$$

or, in the limit,

$$\omega \frac{dr}{dv} = \frac{1}{1 - v^2/c^2}, \qquad (7)$$

and on integration, taking the constant to make $v(0) = 0$:

$$v = c \tanh \omega r/c \qquad (8)$$

By inserting this value in (4), we obtain as our final equations:

$$\begin{aligned} \Theta' &= \Theta \cosh \omega r/c - ct \frac{\sinh \omega r/c}{r}, \ r' = r, \\ z' &= z, \ t' = t \cosh \omega r/c - r\Theta \frac{\sinh \omega r/c}{c}, \end{aligned} \qquad (9)$$

From the form of these equations we see that the totality of transformations corresponding to them forms a group; since the identity is given by setting $\omega = 0$, the inverse of the transformation $\omega = \omega'$ by setting $\omega = -\omega_1$, and the resulting of the two transformations $\omega = \omega_1$ and $\omega = \omega_2$ by setting $\omega = \omega_1 + \omega_2$. We further note that r is restricted to quantities small compared with c, we may replace cosh $\omega r/c$ by unity and sinh $\omega r/c$ by $\omega r/c$, and on neglecting the term in r^2/c^2 we obtain (2*a*). This justifies the use of (2*a*) as a first approximation to obtain results for actual rotating systems, for which of course r/c is negligibly small. (Cf. the discussion of Sagnac's experiment given by M. V. Laue, *Relativitätstheorie*, Vol. 1, 1919, p. 125 and by L. Silberstein in *J. Optical Soc. Amer.*, **5**, 1921, p. 291 f.)

While the relation between the primed and unprimed coördinates, given by (9) is formally symmetrical, the corresponding spaces are not interchangeable. In fact, if the unprimed space is a "stationary" system the square of the arc element of the space-time continuum is given by:

$$d\sigma^2 - c^2\,dt^2 \tag{10}$$

where $d\sigma$ is the arc element for three-dimensional Euclidean space, and consequently in our coördinates by

$$dz^2 + dr^2 + r^2\,d\Theta^2 - c^2\,dt^2. \tag{11}$$

The corresponding element in the rotating, primed system is obtained by solving (9) and substituting the values in (11). This, of course, leaves the four-dimensional space unchanged; but the three dimensional section obtained by putting t' equal to a constant is no longer the arc element of Euclidean space, but is given by the expression (in which primes are dropped):

$$\begin{aligned} & dz^2 + r^2 d\Theta^2 + 2r\,dr\,d\Theta\left(\omega t - \frac{ct}{r}\sinh\frac{\omega r}{c}\cosh\frac{\omega r}{c} - \Theta\sinh^2\frac{\omega r}{c}\right) \\ & \quad + dr^2\left(1 - \frac{r^2\omega^2\Theta^2}{c^2} + \omega^2 t^2 + \frac{c^2t^2}{r^2}\sinh^2\frac{\omega r}{c} - \Theta^2\sinh^2\frac{\omega r}{c}\right. \\ & \left. - 4\,\omega\Theta t\sinh^2\frac{\omega r}{c} - 2\,\frac{ct^2\omega}{r}\sinh\frac{\omega r}{c}\cosh\frac{\omega r}{c} - 2\frac{r\Theta^2\omega}{c}\sinh\frac{\omega r}{c}\cosh\frac{\omega r}{c}\right). \end{aligned} \tag{12}$$

To prove that the space with the above element is non-Euclidean, we have merely to show that its Gaussian curvature is not zero. We only calculate it for the case $t' = 0$. For this special case of (12) the curvature is:

$$R = \frac{\frac{\omega^2}{c^2}\left(1 + 2\Theta^2\left(\frac{\omega r}{c} + \sinh\frac{\omega r}{c}\cosh\frac{\omega r}{c}\right)\left(\frac{\omega r}{c}\sinh\frac{\omega r}{c} - \cosh\frac{\omega r}{c}\right)\left(\frac{\cosh\frac{\omega r}{c}}{\frac{\omega r}{c}}\right)\right)}{\left[1 - \Theta^2\left(\frac{\omega r}{c} + \sinh\frac{\omega r}{c}\cosh\frac{\omega r}{c}\right)^2\right]^2} \quad (13)$$

It appears from the calculations just made that the spacial geometry for the rotating system depends on the time and space coördinates of the point considered, and this is at first sight in contradiction with our ordinary ideas of rotating systems. The explanation of this fact is that for a system rotating with respect to a "stationary" system there is no separation into space and time which stands out as the natural one for the entire rotating system, and consequently for this system spacial geometry is only defined when our coördinates have been selected. Since the equations (9) depend on coördinates so chosen that when $t' = 0$, the points on the radius vector $\Theta^1 = 0$ coincide with those on the radius vector $\Theta = 0$, at $t = 0$ in the stationary system, it is based on coördinates "natural" to these space-time points; in the sense that they are those of a stationary system coinciding with the rotating one at the space-time points considered. It will be noticed that for these points the curvature given by (13) is ω^2/c^2. This shows that the curvature of the spacial cross-section at any space-time point in the coördinates "natural" to this point is constant; it is the square of the angular velocity of rotation in radians per light-second.

THE ENERGY LOSSES ACCOMPANYING IONIZATION AND RESONANCE IN MERCURY VAPOR

By JOHN A. ELDRIDGE

DEPARTMENT OF PHYSICS, UNIVERSITY OF WISCONSIN

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Much light has in recent years been thrown upon the constitution of matter, and upon the validity of the Bohr theory of atomic structure by the study of resonating and ionizing collisions of electrons in vapors and gases. There are, however, still a great many questions of a fundamental nature in regard to such phenomena which cannot be answered by the